

LANGLANDS PARAMETERS ASSOCIATED TO SPECIAL MAXIMAL PARAHORIC SPHERICAL REPRESENTATIONS

MANISH MISHRA

ABSTRACT. In this article, we describe the image, under the local Langlands correspondence for tori, of the characters of a torus which are trivial on its Iwahori subgroup. To a representation of a connected reductive quasi-split group \mathbf{G} , having a non-zero vector fixed under a special maximal parahoric subgroup, we associate in a natural way, a twisted semi-simple conjugacy class in a certain subgroup of the dual group $\hat{\mathbf{G}}$. These results generalize well known classical results to the ramified case.

INTRODUCTION

Let k be a non-archimedean local field. Let \mathbf{G} be an unramified connected reductive group defined over k and let π be a smooth, irreducible, admissible representation of \mathbf{G} which is unramified, i.e., there is a hyperspecial subgroup K of $\mathbf{G}(k)$ such that the K -invariant subspace π^K of the space realizing π is non-zero. One can associate a Langlands parameter to the representation π via the following recipe.

The representation π corresponds, in a natural way, to a character of the Hecke algebra $\mathcal{H}(\mathbf{G}(k), K)$. Then via Satake isomorphism, this character corresponds to an unramified character χ of $\mathbf{T}(k)$ for a certain maximal torus \mathbf{T} . This character χ is unique upto the relative Weyl group conjugacy. The character χ , under the local Langlands correspondence for tori, corresponds to a Langlands parameter $\varphi_\chi \in H^1(W_k, \hat{\mathbf{T}})$, where $\hat{\mathbf{T}}$ is the dual torus and W_k is the Weil group. This parameter is induced from a cocycle in $H^1(W_k/I_k, \hat{\mathbf{T}})$, where I_k is the inertia subgroup of W_k . Using this, one can then associate to the parameter φ_χ a semisimple conjugacy class in $\hat{\mathbf{G}} \rtimes (W_k/I_k)$, where $\hat{\mathbf{G}}$ is the complex dual of \mathbf{G} . This semisimple conjugacy class describes the Langlands parameter associated to π .

All these classic results are well known and can be found in [2]. We wish to find analogous statements when \mathbf{G} is not necessarily unramified. Let \mathbf{G} be quasi-split and tamely ramified. Let K be a special maximal parahoric subgroup of \mathbf{G} and let π be a smooth, irreducible, admissible representation of \mathbf{G} which is K -spherical, i.e., $\pi^K \neq 0$. We associate to π , a character χ of $\mathbf{T}(k)$ of a certain maximal torus \mathbf{T} in a similar way as above, using the description of special maximal parahoric

Hecke algebras given in [3]. We show that χ is trivial on the Iwahori subgroup $\mathbf{T}(k)_0$ of $\mathbf{T}(k)$. In Theorem 1, we calculate the image, under the local Langlands correspondence for tori, of all such characters which are trivial on $\mathbf{T}(k)_0$ and show that it is precisely inflation of the cocycles in $H^1(W_k/I_k, \hat{\mathbf{T}}^{I_k})$. In Theorem 2, we show that the orbits in $H^1(W_k/I_k, \hat{\mathbf{T}}^{I_k})$ of the relative Weyl group are in bijection with the semisimple conjugacy classes in $\hat{\mathbf{H}} \rtimes (W_k/I_k)$, where $\hat{\mathbf{H}}$ is a subgroup of $\hat{\mathbf{G}}$ which we describe explicitly.

In [6, Chapter 11], a character of $\mathbf{T}(k)$ is called elementary, if under the Langlands reciprocity map, it corresponds to a cocycle in $H^1(W_k, \hat{\mathbf{T}})$ which is the inflation of a cocycle in $H^1(W_k/I_k, \hat{\mathbf{T}}^{I_k})$. The question of characterizing the elementary characters is a natural one and some partial results are presented in [6, Chapter 11]. Our first theorem answers this question by showing that a character is elementary if and only if it is trivial on the Iwahori subgroup $\mathbf{T}(k)_0$.

1. NOTATIONS

Let k be a non-archimedean local field. Let \mathbf{G} be a connected reductive group defined over k , which is quasi-split and split over a tamely ramified extension. We denote $\mathbf{G}(k)$ by G and likewise for other algebraic groups. Let K be a special maximal parahoric subgroup of G corresponding to a special vertex ν in the Bruhat-Tits building $\mathcal{B}(G_{ad})$. Let \mathbf{A} denote a maximal split k -torus whose corresponding apartment in $\mathcal{B}(G_{ad})$ contains ν . Let $\mathbf{T} = Z_{\mathbf{G}}\mathbf{A}$, the centralizer of \mathbf{A} in \mathbf{T} . Then \mathbf{T} is a maximal torus in \mathbf{G} since \mathbf{G} is quasi-split. Let W denote the relative Weyl group of \mathbf{G} . Let $\mathcal{H}(G, K)$ be the Hecke algebra of K -bi-invariant compactly supported complex-valued functions on G . Let T_c and T_0 denote respectively the maximal compact subgroup and the Iwahori subgroup of T . Let $\hat{\mathbf{G}}$ denote the complex dual of \mathbf{G} and $\hat{\mathbf{G}}_{ss}$ the set of semisimple elements in $\hat{\mathbf{G}}$. Let $\text{Inn}(\hat{\mathbf{G}})$ be the group of inner automorphisms of $\hat{\mathbf{G}}$. Let $\sigma = \sigma_k$ denotes the Frobenius element in W_k/I_k , where W_k is the Weil group of k and $I = I_k$ is its inertia subgroup. We denote the identity component of an algebraic group \mathcal{G} by \mathcal{G}° .

2. STATEMENT OF THE THEOREMS

Theorem 1. *A character is elementary if and only if it is trivial on the Iwahori subgroup. In other words, we have a commutative diagram:*

$$\begin{array}{ccc}
 \text{Hom}(T, \mathbb{C}^\times) & \xrightarrow[\cong]{LLC} & H^1(W_k, \hat{\mathbf{T}}) \\
 \uparrow & & \uparrow \text{infl} \\
 \text{Hom}(T/T_0, \mathbb{C}^\times) & \xrightarrow[\cong]{} & H^1(W_k/I, \hat{\mathbf{T}}^I),
 \end{array}$$

2

where LLC is the local Langlands correspondence for tori and $infl$ is the inflation homomorphism.

Let $\text{Rep}(G)$ denote the set of equivalent classes of smooth, irreducible, admissible representations of G . Let $\hat{\mathbf{H}} = Z(\hat{\mathbf{G}})^I((\hat{\mathbf{G}}^I)^\circ)$ where $Z(\hat{\mathbf{G}})$ is the center of $\hat{\mathbf{G}}$.

Theorem 2. *The K -spherical representations are in a natural bijection with the semisimple conjugacy classes in $\hat{\mathbf{H}} \rtimes \sigma$ via the local Langlands correspondence for tori. More precisely,*

$$\begin{aligned} \{\pi | \pi \in \text{Rep}(G), \pi^K \neq 0\} &\leftrightarrow \text{Hom}_{\mathbb{C}}(\mathcal{H}(G, K), \mathbb{C}) \\ &\cong \text{Hom}(T/T_0, \mathbb{C}^\times)/W \\ &\cong (\hat{\mathbf{T}}^I)_\sigma/W \\ &\cong \hat{\mathbf{H}}_{ss} \rtimes \sigma / \text{Inn}(\hat{\mathbf{H}}). \end{aligned}$$

3. LANGLANDS CORRESPONDENCE FOR TORI

The following treatment of Local Langlands correspondence for tori can be found in [7].

3.1. The special case of an induced torus. Let $\mathbf{T} = R_{k'/k} \mathbb{G}_m$ be an induced torus, when k' is a finite separable extension of k . Then $T = k'^\times$ and the group of characters $X^*(\mathbf{T})$ is canonically a free \mathbb{Z} -module with basis $W_k/W_{k'}$. From this it follows that $\hat{\mathbf{T}}$ is canonically isomorphic to $\text{ind}_{W_{k'}}^{W_k} \mathbb{C}^\times$. We get

$$\begin{aligned} \text{Hom}(T, \mathbb{C}^\times) &\cong \text{Hom}(k'^\times, \mathbb{C}^\times) \\ (3.1) \quad &\cong \text{Hom}(W_{k'}, \mathbb{C}^\times) \\ &\cong H^1(W_{k'}, \mathbb{C}^\times) \\ (3.2) \quad &\cong H^1(W_k, \text{ind}_{W_{k'}}^{W_k} \mathbb{C}^\times) \\ &\cong H^1(W_k, \hat{\mathbf{T}}). \end{aligned}$$

The isomorphism 3.1 follows by class field theory and the isomorphism 3.2 by Shapiro's lemma.

3.2. The LLC for tori in general.

Theorem. [7, 7.5 Theorem] *There is a unique family of homomorphisms*

$$\varphi_{\mathbf{T}} : \text{Hom}(T, \mathbb{C}^\times) \rightarrow H^1(W_k, \hat{\mathbf{T}})$$

with the following properties:

- (1) $\varphi_{\mathbf{T}}$ is additive functorial in \mathbf{T} , i.e., it is a morphism between two additive functors from the category of tori over k to the category of abelian groups;

- (2) For $\mathbf{T} = R_{k'/k}\mathbb{G}_m$, where k'/k is a finite separable extension, $\varphi_{\mathbf{T}}$ is the isomorphism described in section 3.1.

4. PROOF OF THEOREM 1

Lemma 3. *Let \mathbf{T} be a torus defined over k . Then there exists an isomorphism*

$$\kappa_{\mathbf{T}} : \mathrm{Hom}(T/T_0, \mathbb{C}^\times) \rightarrow H^1(W_k/I, \hat{\mathbf{T}}^I).$$

Moreover, the isomorphism $\kappa_{\mathbf{T}}$ is additive functorial in \mathbf{T} .

Proof. We have by Kottwitz isomorphism [4, sec 7], (see also [3, Prop 1.0.2])

$$\begin{aligned} T/T_0 &\cong ((X^*(\hat{\mathbf{T}}))_I)^\sigma \\ (4.1) \quad &\cong X^*((\hat{\mathbf{T}}^I)_\sigma). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{Hom}(T/T_0, \mathbb{C}^\times) &\cong \mathrm{Hom}(X^*((\hat{\mathbf{T}}^I)_\sigma), \mathbb{C}^\times) \\ (4.2) \quad &\cong (\hat{\mathbf{T}}^I)_\sigma \\ &\cong H^1(W_k/I, \hat{\mathbf{T}}^I). \end{aligned}$$

The isomorphism in equation 4.2 is by Cartier duality. The functoriality of $\kappa_{\mathbf{T}}$ follows from the functoriality of the Kottwitz isomorphism. \square

Remark 4. From the relation

$$T/T_c \cong (X^*(\hat{\mathbf{T}}))_I^\sigma / \text{torsion},$$

one similarly obtains the isomorphism

$$(4.3) \quad \mathrm{Hom}(T/T_c, \mathbb{C}^\times) \cong H^1(W_k/I, (\hat{\mathbf{T}}^I)^\circ).$$

Lemma 5. *Let k'/k be a finite separable extension and put $\mathbf{T} = R_{k'/k}(\mathbb{G}_m)$. Then the isomorphism $\mathrm{Hom}(T/T_0, \mathbb{C}^\times) \cong H^1(W_k/I, \hat{\mathbf{T}}^I)$ obtained from the Kottwitz isomorphism in lemma 3 is the same as the one induced from the Local Langlands correspondence for tori.*

Proof. Since \mathbf{T} is an induced torus, $X^*(\mathbf{T})$ is canonically a free \mathbb{Z} -module with basis $W_k/W_{k'}$. Consequently, $\hat{\mathbf{T}}$ is simply $\mathrm{ind}_{W_{k'}}^{W_k} \mathbb{C}^\times$. Let \mathcal{O} and \mathcal{O}' be the ring of

integers in k and k' respectively. We have,

$$\begin{aligned}
H^1(W_k/I_k, \hat{\mathbf{T}}^{I_k}) &\cong (\hat{\mathbf{T}}^{I_k})_{\sigma_k} \\
&\cong ((\text{ind}_{W_{k'}}^{W_k} \mathbb{C}^\times)^{I_k})_{\sigma_k} \\
&\cong H^1(W_k/I_k, (\text{ind}_{W_{k'}}^{W_k} \mathbb{C}^\times)^{I_k}) \\
(4.4) \quad &\cong H^1(W_{k'}/I_{k'}, \mathbb{C}^\times)
\end{aligned}$$

$$\begin{aligned}
(4.5) \quad &\cong \text{Hom}(k'^\times / \mathcal{O}'^\times, \mathbb{C}^\times) \\
&\cong \text{Hom}(T/T_0, \mathbb{C}^\times).
\end{aligned}$$

Here, the isomorphism in 4.4 is induced by the isomorphism in Shapiro's lemma:

$$\begin{array}{ccc}
H^1(W_k, \text{ind}_{W_{k'}}^{W_k} \mathbb{C}^\times) & \xrightarrow{\sim} & H^1(W_{k'}, \mathbb{C}^\times) \\
\text{infl} \uparrow & & \uparrow \text{infl} \\
H^1(W_k/I_k, (\text{ind}_{W_{k'}}^{W_k} \mathbb{C}^\times)^{I_k}) & \xrightarrow{\sim} & H^1(W_{k'}/I_{k'}, \mathbb{C}^\times).
\end{array}$$

Thus the Local Langlands Correspondence also induces an isomorphism:

$$\varphi_{\mathbf{T}} : \text{Hom}(T/T_0, \mathbb{C}^\times) \cong H^1(W_k/I, \hat{\mathbf{T}}^I).$$

The Kottwitz isomorphism for induced tori is constructed as follows (see [4, Sec. 7.2]): the homomorphism

$$v'_T : T \rightarrow \text{Hom}(X^*(\mathbf{T}), \mathbb{Z}),$$

sending $t \in T$ to the homomorphism

$$\lambda \mapsto \text{ord}(\lambda(t)),$$

induces an isomorphism

$$(4.6) \quad v_T : T/T_0 \cong \text{Hom}(X^*(\mathbf{T})^I, \mathbb{Z})^\sigma.$$

Also, the homomorphism

$$q'_T : X_*(\mathbf{T}) \rightarrow \text{Hom}(X^*(\mathbf{T}), \mathbb{Z}),$$

which sends $\mu \in X_*(T)$ to the homomorphism

$$\lambda \mapsto \langle \lambda, \mu \rangle,$$

induces an isomorphism

$$(4.7) \quad q_T : (X_*(\mathbf{T})_I)^\sigma \cong \text{Hom}(X^*(\mathbf{T})^I, \mathbb{Z})^\sigma.$$

From the equations 4.6 and 4.7, we get the Kottwitz isomorphism for the induced torus \mathbf{T} :

$$w_T : T/T_0 \cong X^*((\hat{\mathbf{T}}^I)_\sigma),$$

where we used the identifications $(X_*(\mathbf{T})_I)^\sigma = (X^*(\hat{\mathbf{T}})_I)^\sigma \cong X^*((\hat{\mathbf{T}}^I)_\sigma)$. This map w_T induces the map $\kappa_{\mathbf{T}} : \text{Hom}(T/T_0, \mathbb{C}^\times) \cong H^1(W_k/I, \hat{\mathbf{T}}^I)$ in lemma 3.

We identify T/T_0 with \mathbb{Z} by the isomorphisms

$$(4.8) \quad T/T_0 \cong k'^\times / \mathcal{O}'^\times \cong \varpi'^{\mathbb{Z}} \cong \mathbb{Z}.$$

Here ϖ' is the uniformizer in k' . Let J be a W_k -stable basis of $X^*(\mathbf{T})$. Choose any $\lambda \in J$ and let $J_\lambda \subset J$ be the orbit of λ in J under the action of I_k . Let $\chi = \sum_{\mu \in J_\lambda} \mu$.

Then $\chi \in X^*(\hat{\mathbf{T}})^{I_k}$.

Let $f \in \text{Hom}(T/T_0, \mathbb{C}^\times)$ and let $c = f(1)$ (under the identification in equation 4.8). Then both $\kappa_{\mathbf{T}}$ and $\varphi_{\mathbf{T}}$ map f to the cocycle $\phi_f \in H^1(W_k/I, \hat{\mathbf{T}}^I)$ defined by $\sigma \mapsto \chi \otimes c$. Thus $\kappa_{\mathbf{T}} = \varphi_{\mathbf{T}}$. This completes the proof of the lemma. \square

Proposition 6. *There is a unique family of homomorphisms*

$$\varphi_{\mathbf{T}} : \text{Hom}(T/T_0, \mathbb{C}^\times) \rightarrow H^1(W_k, \hat{\mathbf{T}}),$$

with the following properties:

- (1) $\varphi_{\mathbf{T}}$ is additive functorial in \mathbf{T} , i.e., it is a morphism between two additive functors from the category of tori over k to the category of abelian groups.
- (2) For $\mathbf{T} = R_{k'/k} \mathbb{G}_m$, where k'/k is a finite Galois extension, $\varphi_{\mathbf{T}}$ is the homomorphism induced from the Local Langlands correspondence for Tori.

Proof. Since the isomorphism $\kappa_{\mathbf{T}}$ in lemma 3 is additive functorial in \mathbf{T} , we thus get an additive functorial family of homomorphisms

$$\varphi_{\mathbf{T}} : \text{Hom}(T/T_0, \mathbb{C}^\times) \longrightarrow H^1(W_k/I, \hat{\mathbf{T}}^I) \xrightarrow{\text{infl}} H^1(W_k, \hat{\mathbf{T}}).$$

This shows existence. To show uniqueness, let \mathbf{T} be a given torus defined over k and let k'/k be a finite Galois extension such that \mathbf{T} is split over k' . Let $\mathbf{T}' = R_{k'/k}(\mathbf{T} \otimes_k k')$. Then \mathbf{T}' is isomorphic to a direct sum of $d = \dim(\mathbf{T})$ tori of the form $R_{k'/k}(\mathbb{G}_m)$ and there is a natural embedding $\mathbf{T} \hookrightarrow \mathbf{T}'$. This gives an embedding $T/T_0 \hookrightarrow T'/T'_0$. By (1), there is a commutative diagram

$$\begin{array}{ccc} \text{Hom}(T/T_0, \mathbb{C}^\times) & \xrightarrow{\varphi_{\mathbf{T}}} & H^1(W_k, \hat{\mathbf{T}}) \\ \uparrow & & \uparrow \\ \text{Hom}(T'/T'_0, \mathbb{C}^\times) & \xrightarrow{\varphi_{\mathbf{T}'}} & H^1(W_k, \hat{\mathbf{T}}'). \end{array}$$

Notice that $\varphi_{\mathbf{T}'}$ is completely determined by (2). Now given $x \in \text{Hom}(T/T_0, \mathbb{C}^\times)$, we can lift it to $x' \in \text{Hom}(T'/T'_0, \mathbb{C}^\times)$. It follows that $\varphi_{\mathbf{T}}(x)$ is the image of $\varphi_{\mathbf{T}'}(x')$ under the vertical arrow on the right, and is hence determined by (1) and (2). \square

Theorem 7. *Let \mathbf{T} be a torus defined over k . Then the local Langlands correspondence for tori induces the isomorphism*

$$\text{Hom}(T/T_0, \mathbb{C}^\times) \cong H^1(W_k/I, \hat{\mathbf{T}}^I).$$

Proof. By the above Proposition, it follows that the homomorphism $\text{Hom}(T/T_0, \mathbb{C}^\times) \rightarrow H^1(W_k, \hat{\mathbf{T}})$, determined by the Kottwitz map must be the same as the one determined by LLC. Therefore, the image of $\text{Hom}(T/T_0, \mathbb{C}^\times)$ under LLC must be the same as the one determined by $\kappa_{\mathbf{T}}$ and which is $H^1(W_k/I, \hat{\mathbf{T}}^I)$. This proves the theorem. \square

5. PROOF OF THEOREM 2

Lemma 8. $\text{Hom}_{\mathbb{C}}(\mathcal{H}(G, K), \mathbb{C}) \cong \text{Hom}(T/T_0, \mathbb{C}^\times)/W$.

Proof. $\mathcal{H}(G, K) \cong \mathbb{C}[T/T_0]^W$ by [3, Theorem 1.0.1]. Therefore

$$\begin{aligned} \text{Hom}_{\mathbb{C}}(\mathcal{H}(G, K), \mathbb{C}) &\cong \text{Hom}_{\mathbb{C}}(\mathbb{C}[T/T_0]^W, \mathbb{C}) \\ &\cong \text{Hom}_{\mathbb{C}}(\mathbb{C}[T/T_0], \mathbb{C})/W \\ &\cong \text{Hom}(T/T_0, \mathbb{C}^\times)/W. \end{aligned}$$

\square

Proposition 9.

$$(\hat{\mathbf{T}}^I)_\sigma/W \cong Z(\hat{\mathbf{G}})^I(\hat{\mathbf{G}}^I)_{ss}^\circ \rtimes \sigma / \text{Inn}(Z(\hat{\mathbf{G}})^I(\hat{\mathbf{G}}^I)^\circ).$$

Proof. Since \mathbf{G} is quasi-split, the inertia group acts on $\hat{\mathbf{G}}$ via a cyclic group (τ) . Let $\phi : \hat{\mathbf{G}}_{sc} \twoheadrightarrow \hat{\mathbf{G}}$ be the simply connected cover of $\hat{\mathbf{G}}$. Let $\hat{\mathbf{Z}}_{sc}$ and $\hat{\mathbf{Z}}$ be the centers of $\hat{\mathbf{G}}_{sc}$ and $\hat{\mathbf{G}}$ respectively. By [5] (remark at the end of cor. 9.12), there is an isomorphism:

$$\hat{\mathbf{Z}}^\tau / \phi \hat{\mathbf{Z}}_{sc} \cong \hat{\mathbf{T}}^\tau / (\hat{\mathbf{T}}^\tau)^\circ.$$

It follow that

$$(5.1) \quad \hat{\mathbf{T}}^\tau = \hat{\mathbf{Z}}^\tau (\hat{\mathbf{T}}^\tau)^\circ.$$

Let \mathbf{W} be the weyl group of $\hat{\mathbf{G}}$. Then \mathbf{W}^τ is the weyl group of $(\hat{\mathbf{G}}^\tau)^\circ$. By [1, lemma 6.5] we have a surjection

$$(5.2) \quad (\hat{\mathbf{T}}^\tau)^\circ \twoheadrightarrow ((\hat{\mathbf{G}}^\tau)^\circ \rtimes \sigma)_{ss} / \text{Inn}((\hat{\mathbf{G}}^\tau)^\circ).$$

By 5.1, this implies

$$(5.3) \quad \hat{\mathbf{T}}^\tau \twoheadrightarrow (\hat{\mathbf{Z}}^\tau (\hat{\mathbf{G}}^\tau)^\circ \rtimes \sigma)_{\text{ss}} / \text{Inn}(\hat{\mathbf{Z}}^\tau (\hat{\mathbf{G}}^\tau)^\circ).$$

Denote the σ -action on an element $g \in \hat{\mathbf{G}}$ by g^σ . Let $s, t \in \hat{\mathbf{T}}^\tau$ be such that there exists $g \in \hat{\mathbf{Z}}^\tau (\hat{\mathbf{G}}^\tau)^\circ$ satisfying $g^{-1}sg^\sigma = t$. Let $\hat{\mathbf{B}}$ be a Borel subgroup of $\hat{\mathbf{G}}$ containing the maximal torus $\hat{\mathbf{T}}$. Write $g = unv$ using Bruhat decomposition, where u, v are in the unipotent radical of $\hat{\mathbf{B}}$ and n is in the normalizer $\hat{\mathbf{N}}$ of $\hat{\mathbf{T}}$. Also, $g = zg_0$ for some $z \in \hat{\mathbf{Z}}^\tau$ and $g_0 \in (\hat{\mathbf{G}}^\tau)^\circ$. Let $g_0 = u_0n_0v_0$ be the Bruhat decomposition of g_0 in $(\hat{\mathbf{G}}^\tau)^\circ$ with respect to the borel $(\hat{\mathbf{B}}^\tau)^\circ$ and mazimal torus $(\hat{\mathbf{T}}^\tau)^\circ$. Then $u = u_0, v = v_0$ and $n = zn_0$. Thus,

$$(5.4) \quad \begin{aligned} g^{-1}sg^\sigma = t &\implies su^\sigma n^\sigma v^\sigma = unvt \\ &\implies su^\sigma s^{-1}sn^\sigma v^\sigma = unt t^{-1}vt \\ &\implies sn^\sigma = nt \\ &\implies sz^\sigma n_0^\sigma = zn_0t. \end{aligned}$$

Let \bar{n}_0 denotes the image of n_0 in \mathbf{W} . Then $n_0^\tau = n_0 \implies \bar{n}_0 \in \mathbf{W}^\tau$. Also 5.4 implies $\bar{n}_0^\sigma = \bar{n}_0$. Thus $\bar{n}_0 \in \mathbf{W}^{\tau, \sigma}$. Then lemma 6.2 in [1] implies that there exists $p \in N_{(\hat{\mathbf{G}}^\tau)^\circ}(\hat{\mathbf{T}}^\tau)^\circ$ such that $p^\sigma = p$ and $n_0 \in p(\hat{\mathbf{T}}^\tau)^\circ$. Let $n_0 = pq$ for some $q \in (\hat{\mathbf{T}}^\tau)^\circ$. Then

$$z^{-1}z^\sigma q^{-1}p^{-1}sp^\sigma q^\sigma = t.$$

Let $r = zq \in \hat{\mathbf{T}}^\tau$. Then $r^{-1}p^{-1}spr^\sigma = t$. Thus $\bar{s} = \bar{t}$ where \bar{s} and \bar{t} represent the class of s and t in $\hat{\mathbf{T}}_\sigma^\tau / \mathbf{W}^{\tau, \sigma} = (\hat{\mathbf{T}}^I)_\sigma / W$. \square

The proof of the bijection

$$\{\pi | \pi \in \text{Rep}(G), \pi^K \neq 0\} \leftrightarrow \text{Hom}_{\mathbb{C}}(\mathcal{H}(\mathbf{G}, K), \mathbb{C}).$$

is identical to the case when \mathbf{G} is unramified and K is hyperspecial [2]. This completes the proof of the theorem 2.

ACKNOWLEDGEMENTS

I would like to thank my advisor Jiu-Kang Yu for his suggestion of this problem and for his careful mentoring throughout. I would also like to thank Sungmun Cho for many helpful suggestions.

REFERENCES

- [1] A. Borel. Automorphic L -functions. In *Automorphic forms, representations and L -functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979.
- [2] P. Cartier. Representations of p -adic groups: a survey. In *Automorphic forms, representations and L -functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part

- 1, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.
- [3] T.J. Haines and S. Rostami. The Satake isomorphism for special maximal parahoric Hecke algebras. *Represent. Theory*, 14:264–284, 2010.
 - [4] Robert E. Kottwitz. Isocrystals with additional structure. II. *Compositio Math.*, 109(3):255–339, 1997.
 - [5] R. Steinberg. Endomorphisms of Linear Algebraic Groups. *Mem. Amer. Math. Soc.*, 80, 1968.
 - [6] Rainer Weissauer. *Endoscopy for $\mathrm{GSp}(4)$ and the cohomology of Siegel modular threefolds*, volume 1968 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
 - [7] Jiu-Kang Yu. On the local Langlands correspondence for tori. In *Ottawa lectures on admissible representations of reductive p -adic groups*, volume 26 of *Fields Inst. Monogr.*, pages 177–183. Amer. Math. Soc., Providence, RI, 2009.

E-mail address: mmishra@math.purdue.edu

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY 150 N. UNIVERSITY STREET, WEST LAFAYETTE, IN 47907-2067